REMARKS

Claims 1, 3, 4, 6, and 7 remain in this application. Claims 2, 5, and 8 have been cancelled. Claims 1, 4 and 6 have been amended.

Claims 6 and 7 were rejected under Section 112, first paragraph, as failing to comply with the enablement requirement. Specifically, the examiner stated that it is unclear from the specification how the leaf spring arms can be converted from pivoting arms to fixed arms during roll motion of the vehicle. Applicant has amended claim 6 to make clear that "the longitudinal suspension arms upon which the air bags are mounted act as beams which are pivotally mounted at their one ends to the frame or chassis of the vehicle during normal vehicle motion and which are caused to act as beams which are fixed or tending towards 'encastre' at their pivotally connected ends by the anti-roll means during roll motion of the vehicle." Applicant submits that claim 6 as amended is more clear and meets the enabling requirement. Further, Applicant submits that claims 6 and 7 are enabled by the specification at page 14, line 1 through page 15, line 3.

Claims 1 - 8 were rejected under Section 103(a) as being unpatentable over McJunkin, Jr. (U.S. Patent No. 3,711,079, hereinafter "McJunkin") in view of Wilson (U.S. Patent No. 5,938,221, hereinafter "Wilson"). Applicant respectfully traverses this rejection.

The present invention as found in the amended claims is directed to an air suspension anti-roll stabilization system in which an anti-roll means is connected rigidly to a pair of longitudinal leaf spring suspension arms <u>at or adjacent connection points at which one end of each suspension arm is pivotally mounted to a vehicle frame or chassis</u>. The anti-roll means is connected between the connection points such that it adds transverse, torsional stiffness to the suspension arms <u>at or close to the connection points during vehicle roll</u>. In a preferred arrangement, the anti-roll means comprises a bar connected directly transversely between the connection points such that it acts on the connection points adjacent the pivotal connections points of the suspension arms to the vehicle frame or

chassis. Basis for the amendment of claim 1 can be found at page 10, lines 12 - 25, page 12, lines 4 - 8, and page 15, lines 14 - 20 of the application as filed.

In particular, Figure 7C shows that the torque created by the torsional stiffness mentioned above generates opposed moments C and D which reduce the spring deflection as would occur with a fixed ended beam, rather than in Figure 7B where a pin-jointed beam bending moment is shown. This ability to increase the bending moment stiffness of the leaf spring arm during roll of the vehicle, which is the result of the subject suspension of the present application, creates a vastly superior air suspension system, in that the geometry of the inventive system provides a much softer ride under normal straight ride conditions and high stability under dynamic roll conditions.

In contrast to the present invention, the stabilizing bar taught by McJunkin is a generally U-shaped bar (22, 23, 33) of which a central portion (33) supplies a torque to resist roll of the vehicle (column 3, lines 13 - 23). The central portion (33) of the bar is positioned parallel to and adjacent the vehicle axle being secured through rearwardly directed legs (22, 23) which are secured to respective suspension arms (12, 13) extending in the longitudinal directions of said arms. It should be noted that the primary function of the stabilizing bar (22, 23, 33) is to dampen any undesirable deflections of the suspension arms through the resistance to deflection of said arms (22, 23) of the bar in the longitudinal directions of the suspension arms (12, 13) (column 2, line 64 through column 4, line 6). In other words, in McJunkin, the central portion (33) of the stabilization bar, namely the portion of the bar that spans between the suspension arms, is connected adjacent the axle (25, 26) and distant from the points (eg., 15, 17) at which the suspension arms are pivotally connected to the frame. Contrarily, in the present invention as claimed in claim 1, the "anti-roll means is connected rigidly to the pair of longitudinal leaf spring suspension arms at or adjacent connection points at which the one end of each suspension arm is pivotally mounted to the frame or chassis."

In McJunkin, under vehicle roll conditions where the suspension arms (12, 13) are caused to deflect in opposite directions, it can be seen that the central portion (33) of the

stabilizing bar adds transverse, torsional stiffness to the suspension arms at or close to the connection points (36, 37) adjacent to the axle rather than to the connection points (34, 35) adjacent to the points by which the suspension arms are pivotally mounted to the vehicle frame or chassis. Consequently, in order to provide a similar degree of torsional stiffness to the suspension arms as provided by the arrangement of the present invention, it would be necessary to make the thickness of the central portion (33) of the stabilizing bar considerably thicker. This would either necessitate making the arms (22, 23) equally thick, thus making it more difficult to mount them to the suspension arms, or providing arms of a narrower gauge to the central portion, which would create weakness points where the change in thicknesses occurred between the central portion and the legs. In any event, in the arrangement of McJunkin, it requires a proportionately greater deflection in opposite directions of the suspension arms to generate the same degree of twisting (torsional) movement in the stabilizing bar than is the case in the present invention, thereby adding support to the foregoing point that the stabilizing bar of the present invention need not be made as thick as that required by McJunkin.

A further advantage offered by the preferred arrangement of the present invention is that by having the anti-roll bar connected transversely between the connection points, oppositely directed deflections of the suspension arms not only causes twisting of the anti-roll bar but also seeks to stretch it, thereby further resisting rolling of the vehicle. In contrast, in McJunkin the arrangement of the stabilizing bar adjacent and parallel the axle results in little or no stretching of the bar during vehicle roll.

The arrangement disclosed in McJunkin is by its nature a distinct design choice with no suggestion being provided therein that the stabilizing bar could be arranged with the central portion distanced away from the vehicle axle in order to apply torsional stiffness to the suspension arms at some location other than adjacent the axle. As such, it cannot now be concluded that such an alternative arrangement is obvious as a means of objecting to the present invention, particularly in view of the considerable amount of time that has elapsed since McJunkin was published in 1973.

Applicant has also attached copies of pages from two textbooks which provide definitions and examples of beams with pin-jointed ends (normal ride conditions of the leaf springs) and fixed or encastre ends (roll conditions). The first textbook reference is G. H. Ryder, *Strength of Materials* 72-73, 152-153, 178-179 (2d ed., Cleaver-Hume Press Ltd. 1958). The second textbook reference is Raymond J. Roark, *Formulas for Stress and Strain* 102-105 (3d ed., McGraw-Hill Book Company, Inc. 1954).

For these reasons, McJunkin does not teach or suggest the features of the presently claimed invention. Further, there is nothing in the teaching of Wilson which would enable one skilled in the art to overcome the aforementioned shortcomings in McJunkin when contrasted with the present invention as now claimed. Therefore, applicant respectfully requests that the Section 103(a) rejection of claim 1 over McJunkin, Jr. in view of Wilson be withdrawn.

This amendment and request for reconsideration is felt to be fully responsive to the comments and suggestions of the examiner and to present the claims in condition for allowance. Favorable action is requested.

Respectfully submitted,

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STRENGTH
of
MATERIALS

By

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Preface to the Second Edition

Extensive additions have been made to this new edition, either to bring it up to date with new developments, to improve the original presentation, or to keep up with the widening scope of examination syllabuses. The emphasis on basic principles and interpretation of the underlying physical behaviour is however maintained and extended to the new material.

There are additions on Material Testing and Experimental Methods, and the effects of stress concentrations in members under tensile, bending, and twisting loads are discussed in their relevant contexts. Extensions have been made to the elastic theory in the fields of strain analysis, with particular reference to resistance strain gauge practice; torsion of thin-walled and cellular tubes and open sections; beams on elastic foundations; and strut analysis by the energy method. Developments in the plastic yielding of steel are given prominence, with a new chapter on the Plastic Theory of Bending, and sections on the plastic yielding of shafts and of tubes under pressure.

The number and scope of illustrative examples and of problems to be worked is now considerably increased, and additional references have been given at the ends of chapters, particularly to works on the subject of a practical nature.

G. H. Ryder

March, 1957

From the Original Preface

This book sets out to cover in one volume the whole of the work required up to Final Degree standard in Strength of Materials. The only prior knowledge assumed is of elementary Applied Mechanics and Calculus. Consequently, it should prove of value to students preparing for a Higher National Certificate and Professional Institution examinations, as well as those following a Degree course. The contents are based on the Syllabus of the University of London, with certain additions.

The main aim has been to give a clear understanding of the principles underlying engineering design, and a special effort has been made to indicate the shortest analysis of each particular problem. Each chapter, starting with assumptions and theory, is complete in itself and is built

portion is clockwise, and on the right portion anticlockwise. This is referred to as *sagging* bending moment since it tends to make the beam concave upwards at AA. Negative bending moment is termed *hogging*.

A *bending moment diagram* is one which shows the variation of bending moment along the length of the beam.

5.3. **Types of Load.** A beam is normally horizontal, the loads being vertical, other cases which occur being looked upon as exceptions.

A *concentrated* load is one which is considered to act at a point, although in practice it must really be distributed over a small area.

A *distributed* load is one which is spread in some manner over the length of the beam. The rate of loading w is quoted as "lb./ft. run" or "tons/ft. run," and may be uniform, or may vary from point to point along the beam.

Defines temp.

5.4. **Types of Support.** A *simple* or *free* support is one on which the beam is rested, and which exerts a reaction on the beam. Normally the reaction will be considered as acting at a point, though it may be distributed along a length of beam in a similar manner to a distributed load.

A *built-in* or *encastré* support is frequently met with, the effect being to fix the direction of the beam at the support. In order to do this the support must exert a "fixing" moment M and a reaction R on the beam (Fig. 75). A beam thus fixed at one end is called a *cantilever*; when fixed at both ends the reactions are not statically determinate, and this case will be dealt with later (Chapter X).

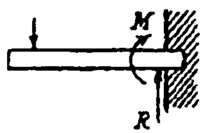

Fig. 75

In practice it is not usually possible to obtain perfect fixing, and the "fixing" moment applied will be related to the angular movement at the support. When in doubt about the rigidity (e.g. a riveted joint), it is "safer" to assume that the beam is freely supported.

5.5. **Relations between w, F, and M.** Fig. 76 shows a short length

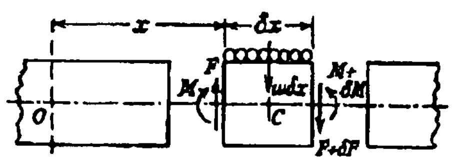

Fig. 76

δx imagined to be a "slice" cut out from a loaded beam at a distance x from a fixed origin O.

Let the shearing force at the section x be F, and at $x+\delta x$ be $F+\delta F$. Similarly, the bending moment is M at x, and $M+\delta M$ at $x+\delta x$. If w is the mean rate of loading on the length δx, the total load is $w\delta x$, acting approximately (exactly, if uniformly distributed) through the centre C. The element must be in equilibrium under the action of these forces and couples, and the following equations are obtained.

Taking moments about C:

$$M + F.\delta x/2 + (F+\delta F)\delta x/2 = M + \delta M$$

Neglecting the product $\delta F.\delta x$, and taking the limit, gives

$$F = dM/dx \qquad (1)$$

Resolving vertically

$$w\delta x + F + \delta F = F$$

or

$$w = -dF/dx \qquad (2)$$

$$= -d^2M/dx^2 \quad \text{from (1)} \qquad (3)$$

From equation (1) it can be seen that, if M is varying continuously, zero shearing force corresponds to maximum or minimum bending moment, the latter usually indicating the greatest value of negative bending moment. It will be seen later, however, that "peaks" in the bending moment diagram frequently occur at concentrated loads or reactions, and are not then given by $F = dM/dx = 0$, although they may represent the greatest bending moment on the beam. Consequently it is not always sufficient to investigate the points of zero shearing force when determining the maximum bending moment.

At a point on the beam where the type of bending is changing from sagging to hogging, the bending moment must be zero, and this is called a point of *inflection* or *contraflexure*.

By integrating equation (1) between two values of $x = a$ and b, then

$$M_b - M_a = \int_a^b F dx$$

showing that the increase in bending moment between two sections is given by the area under the shearing force diagram.

Similarly, integrating equation (2)

$$F_a - F_b = \int_a^b w dx$$

= the area under the load distribution diagram.

Integrating equation (3) gives

$$M_a - M_b = \iint_a^b w dx.dx$$

These relations prove very valuable when the rate of loading cannot

CHAPTER IX

Deflection of Beams

9.1. Strain Energy due to Bending. Consider a short length of beam δx, under the action of a bending moment M. If f is the bending stress on an element of the cross-section of area δA at a distance y from the neutral axis, the strain energy of the length δx is given by

$$\delta U = \int (f^2/2E) \times \text{volume} \quad \text{(Para. 1.9)}$$
$$= \delta x \int f^2 . dA/2E$$
$$= (\delta x/2E) \int M^2 y^2 dA/I^2$$

But $\int y^2 . dA = I$

hence $\delta U = (M^2/2EI)\delta x$

For the whole beam:

$$U = \int M^2 . dx/2EI$$

The product EI is called the ***Flexural Rigidity*** of the beam.

EXAMPLE 1. *A simply supported beam of length l carries a concentrated load W at distances of a and b from the two ends. Find expressions for the total strain energy of the beam and the deflection under the load.*

beam with pin jointed ends.

The integration for strain energy can only be applied over a length of beam for which a continuous expression for M can be obtained. This usually implies a separate integration for each section between two concentrated loads or reactions.

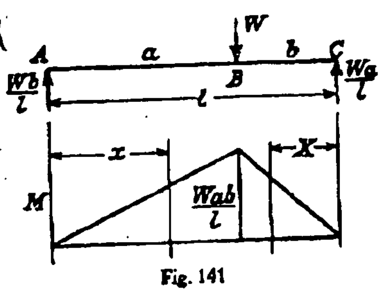

Fig. 141

Referring to Fig. 141, for the section AB,

$$M = (Wb/l)x$$

$$U_a = \int_0^a \frac{W^2b^2x^2}{2l^2EI} . dx$$
$$= \frac{W^2b^2}{2l^2EI}\left[\frac{x^3}{3}\right]_0^a$$
$$= W^2a^3b^2/6EIl^2$$

Similarly, by taking a variable X measured from C

$$U_b = \int_0^b \frac{W^2a^2X^2}{2l^2EI} . dX = W^2a^2b^3/6EIl^2$$

$$\text{Total } U = U_a + U_b = (W^2a^2b^2/6EIl^2)(a+b)$$
$$= W^2a^2b^2/6EIl$$

But, if δ is the deflection under the load, the strain energy must equal the work done by the load (gradually applied), i.e.

$$\tfrac{1}{2}W\delta = W^2a^2b^2/6EIl$$
$$\therefore \delta = Wa^2b^2/3EIl$$

For a central load, $a = b = l/2$, and

$$\delta = (W/3EIl)(l^2/4)(l^2/4)$$
$$= Wl^3/48EI$$

It should be noted that this method of finding deflection is limited to cases where only one concentrated load is applied (i.e. doing work), and then only gives the deflection under the load. A more general application of strain energy to deflection is found in Castigliano's theorem (Para. 11.4).

EXAMPLE 2. *Compare the strain energy of a beam, simply supported at its ends and loaded with a uniformly distributed load, with that of the same beam centrally loaded and having the same value of maximum bending stress.* (U.L.)

If l is the span and EI the flexural rigidity, then for a uniformly distributed load w, the end reactions are $wl/2$, and at a distance x from one end

$$M = (wl/2)x - wx^2/2$$
$$= (wx/2)(l - x)$$

$$U_1 = \int_0^l \frac{w^2x^2(l-x)^2dx}{4 \times 2EI}$$
$$= \frac{w^2}{8EI}\int_0^l (l^2x^2 - 2lx^3 + x^4)dx$$
$$= (w^2l^5/8EI)(\tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5})$$
$$= w^2l^5/240EI \qquad \text{(i)}$$

For a central load of W,

$$U_2 = \tfrac{1}{2}W\delta$$
$$= W^2l^3/96EI \qquad \text{(ii)}$$

see also Example 1.

Maximum bending stress $= \hat{M}/Z$, and for a given beam depends on the maximum bending moment.

Equating maximum bending moments,

$$wl^2/8 = Wl/4 \quad \text{(Chap. 5)} \qquad \text{(iii)}$$
$$\therefore wl = 2W$$

$$\text{Ratio } U_1/U_2 = (w^2l^5/240)(96/W^2l^3) \quad \text{from (i) and (ii)}$$
$$= (96/240)(w^2l^2/W^2)$$
$$= (96/240)4 \quad \text{from (iii)}$$
$$= 8/5$$

Built-In and Continuous Beams

10.1. Moment-Area Method for Built-in Beams. A beam is said to be built-in or encastre when both its ends are rigidly fixed so that the slope remains horizontal. Usually also the ends are at the same level.

It follows from the moment-area method (Para. 9.5) that, since the change of slope from end to end and the intercept x are both zero

$$\Sigma A = 0 \qquad (1)$$

and

$$\Sigma A\bar{x} = 0 \qquad (2)$$

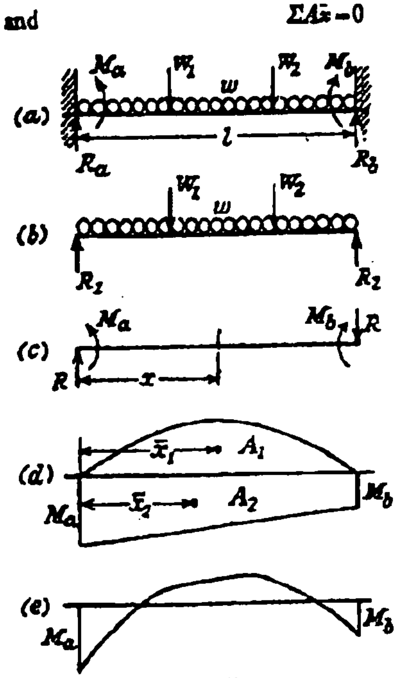

Fig. 168

It will be found convenient to show the bending moment diagram due to any loading such as Fig. 168(a) as the algebraic sum of two parts, one due to the loads, treating the beam as simply supported (Fig. 168(b)), and the other due to the end moments introduced to bring the slopes back to zero (Fig. 168(c)).

The area and end reactions obtained if freely supported will be referred to as the *free moment diagram* and the *free reactions*, A_1, R_1 and R_2 respectively.

The fixing moments at the ends are M_a and M_b, and in order to maintain equilibrium when M_a and M_b are unequal, the reactions $R = (M_a - M_b)/l$ are introduced, being upwards at the left-hand end and downwards at the right-hand end. Due to M_a, M_b, and R, the bending moment at a distance x from the left-hand end

$$= -M_a + Rx = -M_a + [(M_a - M_b)/l]x.$$

This gives a straight line going from a value $-M_a$ at $x = 0$ to $-M_b$ at $x = l$, and hence the *fixing moment diagram*, A_2 (Fig. 168 (d)).

For downward loads, A_1 is a positive area (sagging B.M.), and A_2 a negative area (hogging B.M.) consequently the equations (1) and (2) reduce to

$$A_1 = A_2 \qquad (1)$$

and

$$A_1\bar{x}_1 = A_2\bar{x}_2 \quad \text{(numerically)} \qquad (2)$$

i.e. **Area of free moment diagram = Area of fixing moment diagram**

and **Moments of areas of free and fixing diagrams are equal.**

It may be necessary to break down the areas still further to obtain convenient triangles and parabolas.

These two equations enable M_a and M_b to be found, and the total reactions at the ends are

$$R_a = R_1 + R$$
$$= R_1 + (M_a - M_b)/l$$

and

$$R_b = R_2 - R$$
$$= R_2 - (M_a - M_b)/l$$

Finally, the combined bending moment diagram is shown in Fig. 168(e) as the algebraic sum of the two components.

EXAMPLE 1. *Obtain expressions for the maximum bending moment and deflection of a beam of length l and flexural rigidity EI, fixed horizontally at both ends, carrying a load W (a) concentrated at mid-span, (b) uniformly distributed over the whole beam.*

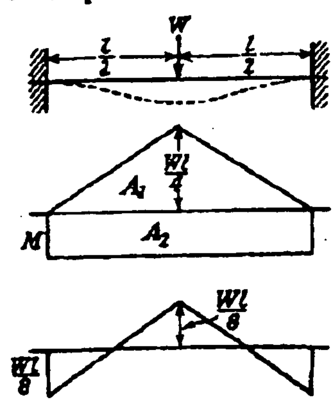

Fig. 169

(a) By symmetry $M_a = M_b = M$, say (Fig. 169).

The free moment diagram is a triangle with maximum ordinate $Wl/4$ (Chap. V).

$$\therefore \quad \text{Area } A_1 = \tfrac{1}{2}(Wl/4)l$$
$$= Wl^2/8$$
$$\text{Area } A_2 = Ml$$

Equating $A_1 = A_2$ from (1), gives

$$M = Wl/8$$

The combined bending moment diagram is therefore as shown in the lower diagram, Fig. 169, and the maximum bending moment is $Wl/8$, occurring at the end (hogging), and the centre (sagging).

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PREFACE TO THE THIRD EDITION

As in the first revision, new data have been added, and tables of formulas and coefficients have been amplified. Some of the more important changes are as follows:

In Chap. 8 (Beams) the discussion of shear lag has been rewritten to include the results of recent investigations, and in Table VIII formulas for circular arches have been added. In Chap. 10 (Flat Plates) the table of stress and deflection coefficients has been expanded to cover a number of additional cases and to include coefficients for edge slope; also a table of coefficients for rectangular plates with large deflection has been added. In Chap. 11 (Columns) Table XI has been revised to bring it in line with current specifications. In Chap. 12 (Pressure Vessels) Table XIII has been extensively revised and amplified, and the former example of stress calculation for thin vessels has been replaced by one that illustrates the use of the new formulas and provides comparison with experimental results. Table XVII (Factors of Stress Concentration) has been extended to include factors based on the important work of Neuber.

In addition, miscellaneous formulas and data believed to be of value have been introduced in appropriate chapters, and the reference lists have been revised and extended.

The literature pertaining to applied mechanics and elasticity has grown to such proportions that it is manifestly impossible to include more than a small fraction of it in a single volume, even by reference. Those working in the field will of course be familiar with the important sources of published material; others will be able to gain some idea of where to seek additional information from the references given in this book and from the available bibliographies and digests, particularly from "Applied Mechanics Reviews," published monthly by the American Society of Mechanical Engineers, and from the "Technical Data Digest," published by the Central Air Documents Office.

Again the author wishes to thank the many readers to whom he is indebted for suggestions and for help in detecting errors and omissions. In particular he wishes to make grateful acknowledgment to Prof. Eric Reissner of the Massachusetts Institute of

11 = Symmetrical spring under normal deflection.

TABLE III.—SHEAR, MOMENT, AND DEFLECTION FORMULAS FOR BEAMS.—(Continued)

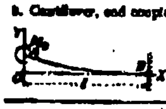
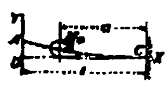
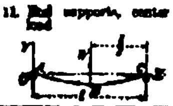
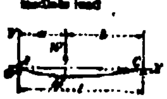
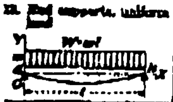
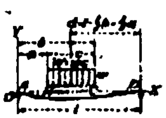
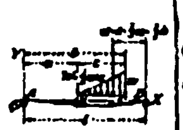
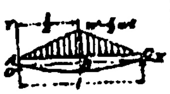
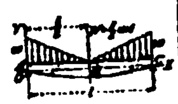
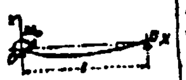
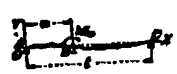
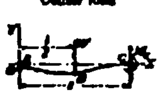
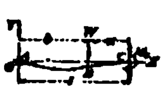
Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Bending moment M and maximum bending moment	Deflection y, maximum deflection, and end slope θ
9. Cantilever, end couple 	$R_1 = 0$ $V = 0$	$M = M_0$ Max $M = M_0$ (A to B)	$y = \frac{1}{2}\frac{M_0}{EI}(l^2 - 2lx + x^2)$ Max $y = +\frac{1}{2}\frac{M_0 l^2}{EI}$ at A $\theta = -\frac{M_0 l}{EI}$ at A
10. Cantilever, intermediate couple 	$R_1 = 0$ $V = 0$	(A to B) $M = 0$ (B to C) $M = M_0$ Max $M = M_0$ (B to C)	(A to B) $y = \frac{M_0 a}{EI}\left(l - \frac{1}{2}a - x\right)$ (B to C) $y = \frac{1}{2}\frac{M_0}{EI}[(x - l + a)^2 - 2a(x - l + a) + a^2]$ Max $y = \frac{M_0 a}{EI}\left(l - \frac{1}{2}a\right)$ at A $\theta = -\frac{M_0 a}{EI}$ (A to B)
11. End supports, center load 	$R_1 = +\frac{1}{2}W$ $R_2 = +\frac{1}{2}W$ (A to B) $V = +\frac{1}{2}W$ (B to C) $V = -\frac{1}{2}W$	(A to B) $M = +\frac{1}{2}Wx$ (B to C) $M = +\frac{1}{2}W(l - x)$ Max $M = +\frac{1}{4}Wl$ at B	(A to B) $y = -\frac{1}{48}\frac{W}{EI}(3l^2x - 4x^3)$ Max $y = -\frac{1}{48}\frac{Wl^3}{EI}$ at B $\theta = -\frac{1}{16}\frac{Wl^2}{EI}$ at A, $\theta = +\frac{1}{16}\frac{Wl^2}{EI}$ at C
12. End supports, intermediate load 	$R_1 = +W\frac{b}{l}$ $R_2 = +W\frac{a}{l}$ (A to B) $V = +W\frac{b}{l}$ (B to C) $V = -W\frac{a}{l}$	(A to B) $M = +W\frac{b}{l}x$ (B to C) $M = +W\frac{a}{l}(l - x)$ Max $M = +W\frac{ab}{l}$ at B	(A to B) $y = -\frac{Wbx}{6EIl}[2l(l - x) - b^2 - (l - x)^2]$ (B to C) $y = -\frac{Wa(l - x)}{6EIl}[2lb - b^2 - (l - x)^2]$ Max $y = -\frac{Wab}{27EIl}(a + 2b)\sqrt{3a(a + 2b)}$ at $x = \sqrt{\frac{1}{3}a(a + 2b)}$ when $a > b$ $\theta = -\frac{1}{6}\frac{W}{EI}\left(bl - \frac{b^3}{l}\right)$ at A; $\theta = +\frac{1}{6}\frac{W}{EI}\left(2bl + \frac{b^3}{l} - 3b^2\right)$ at C
13. End supports, uniform load 	$R_1 = +\frac{1}{2}W$ $R_2 = +\frac{1}{2}W$ $V = \frac{1}{2}W\left(1 - \frac{2x}{l}\right)$	$M = \frac{1}{2}W\left(x - \frac{x^2}{l}\right)$ Max $M = +\frac{1}{8}Wl$ at $x = \frac{1}{2}l$	$y = -\frac{1}{24}\frac{Wx}{EIl}(l^3 - 2lx^2 + x^3)$ Max $y = -\frac{5}{384}\frac{Wl^3}{EI}$ at $x = \frac{1}{2}l$ $\theta = -\frac{1}{24}\frac{Wl^2}{EI}$ at A $\theta = +\frac{1}{24}\frac{Wl^2}{EI}$ at B
14. End supports, partial uniform load 	$R_1 = W\frac{d}{l}$ $R_2 = \frac{W}{l}\left(a + \frac{1}{2}e\right)$ (A to B) $V = R_1$ (B to C) $V = R_1 - W\frac{x - a}{e}$ (C to D) $V = R_1 - W$	(A to B) $M = R_1x$ (B to C) $M = R_1x - W\frac{(x - a)^2}{2e}$ (C to D) $M = R_1x - W(x - \frac{1}{2}a - \frac{1}{2}b)$ Max $M = W\frac{d}{l}\left(a + \frac{de}{2l}\right)$ at $x = a + \frac{de}{l}$	(A to B) $y = \frac{1}{48EI}\left\{8R_1(x^3 - l^2x) + Wx\left[\frac{8d^3}{l} - \frac{2be^2}{l} + \frac{e^3}{l} + 2e^2\right]\right\}$ (B to C) $y = \frac{1}{48EI}\left\{8R_1(x^3 - l^2x) + Wx\left[\frac{8d^3}{l} - \frac{2be^2}{l} + \frac{e^3}{l} + 2e^2\right] - 2W\frac{(x - a)^4}{e}\right\}$ (C to D) $y = \frac{1}{48EI}\left\{8R_1(x^3 - l^2x) + Wx\left[\frac{8d^3}{l} - \frac{2be^2}{l} + \frac{e^3}{l}\right] - 8W(x - \frac{1}{2}a - \frac{1}{2}b)^3 + W(2be^2 - e^3)\right\}$ $\theta = \frac{1}{48EI}\left[-8R_1l^2 + W\left(\frac{8d^3}{l} - \frac{2be^2}{l} + \frac{e^3}{l} + 2e^2\right)\right]$ at A; $\theta = \frac{1}{48EI}\left[16R_1l^2 - W\left(24d^2 - \frac{8d^3}{l} + \frac{2be^2}{l} - \frac{e^3}{l}\right)\right]$ at D
15. End supports, triangular load 	$R_1 = \frac{1}{3}W$ $R_2 = \frac{2}{3}W$ $V = W\left(\frac{1}{3} - \frac{x^2}{l^2}\right)$	$M = \frac{1}{3}W\left(x - \frac{x^3}{l^2}\right)$ Max $M = 0.128Wl$ at $x = l\left(\frac{\sqrt{3}}{3}\right) = 0.5774l$	$y = -\frac{1}{180}\frac{Wx}{EIl^2}(3x^4 - 10l^2x^2 + 7l^4)$ Max $y = -0.01304\frac{Wl^3}{EI}$ at $x = 0.5193l$ $\theta = -\frac{7}{180}\frac{Wl^2}{EI}$ at A; $\theta = +\frac{8}{180}\frac{Wl^2}{EI}$ at B
16. End supports, partial triangular load 	$R_1 = W\frac{d}{l}$ $R_2 = W\frac{l - d}{l}$ (A to B) $V = +R_1$ (B to C) $V = R_1 - \left(\frac{x - a}{e}\right)^2 W$ (C to D) $V = R_1 - W$	(A to B) $M = R_1x$ (B to C) $M = R_1x - W\frac{(x - a)^3}{3e^2}$ (C to D) $M = R_1x - \frac{1}{3}W(3x - a - 2b)$ Max $M = W\frac{d}{l}\left(a + \frac{2}{3}e\sqrt{\frac{d}{l}}\right)$ at $x = a + e\sqrt{\frac{d}{l}}$	(A to B) $y = \frac{1}{6EI}\left\{R_1(x^3 - l^2x) + Wx\left[\frac{d^3}{l} + \frac{1}{6}e^2\left(1 - \frac{b}{l}\right) + \frac{17}{270}\frac{e^3}{l}\right]\right\}$ (B to C) $y = \frac{1}{6EI}\left[R_1(x^3 - l^2x) - \frac{1}{10}W\frac{(x - a)^5}{e^2} + Wx\left(\frac{d^3}{l} + \frac{1}{6}e^2 - \frac{1}{6}e^2\frac{b}{l} + \frac{17}{270}\frac{e^3}{l}\right)\right]$ (C to D) $y = \frac{1}{6EI}\left\{R_1(x^3 - l^2x) - W\left[\left(x - \frac{1}{3}a - \frac{2}{3}b\right)^3 - x\frac{d^3}{l} - \frac{1}{6}xe^2\left(1 - \frac{b}{l}\right) + \frac{17}{270}e^3\left(1 - \frac{x}{l}\right)\right]\right\}$ $\theta = \frac{1}{6EI}\left[-R_1l^2 + W\left(\frac{d^3}{l} + \frac{1}{6}e^2 + \frac{17}{270}\frac{e^3}{l} - \frac{1}{6}\frac{e^2b}{l}\right)\right]$ at A $\theta = \frac{1}{6EI}\left[2R_1l^2 + W\left(\frac{d^3}{l} + \frac{17}{270}\frac{e^3}{l} - \frac{1}{6}\frac{e^2b}{l} - 3d^2\right)\right]$ at D

TABLE III.—SHEAR, MOMENT, AND DEFLECTION FORMULAS FOR BEAMS.—(*Continued*)

Loading, support, and reference number	Reactions R_1 and R_2, vertical shear V	Bending moment M and maximum bending moment	Deflection y, maximum deflection, and end slope θ
17. End supports, triangular load 	$R_1 = \frac{1}{2}W$ $R_2 = \frac{1}{2}W$ (A to B) $V = \frac{1}{2}W\left(1 - 4\frac{x^2}{l^2}\right)$ (B to C) $V = -\frac{1}{2}W\left(1 - 4\frac{(l-x)^2}{l^2}\right)$	(A to B) $M = \frac{1}{6}W\left(3x - 4\frac{x^3}{l^2}\right)$ (B to C) $M = \frac{1}{6}W\left[3(l-x) - 4\frac{(l-x)^3}{l^2}\right]$ Max $M = \frac{1}{6}Wl$ at B	(A to B) $y = \frac{1}{6}\frac{Wx}{EIl}\left(\frac{1}{2}lx^2 - \frac{1}{5}x^3 - \frac{5}{16}l^3\right)$ Max $y = -\frac{1}{60}\frac{Wl^3}{EI}$ at B $\theta = -\frac{5}{96}\frac{Wl^2}{EI}$ at A; $\theta = +\frac{5}{96}\frac{Wl^2}{EI}$ at C
18. End supports, triangular load 	$R_1 = \frac{1}{2}W$ $R_2 = \frac{1}{2}W$ (A to B) $V = \frac{1}{2}W\left(\frac{l-2x}{l}\right)^2$ (B to C) $V = -\frac{1}{2}W\left(\frac{2x-l}{l}\right)^2$	(A to B) $M = \frac{1}{2}W\left(x - 2\frac{x^2}{l} + \frac{4}{3}\frac{x^3}{l^2}\right)$ (B to C) $M = \frac{1}{2}W\left[(l-x) - 2\frac{(l-x)^2}{l} + \frac{4}{3}\frac{(l-x)^3}{l^2}\right]$ Max $M = \frac{1}{12}Wl$ at B	(A to B) $y = \frac{1}{12}\frac{W}{EI}\left(x^3 - \frac{x^4}{l} + \frac{2}{5}\frac{x^5}{l^2} - \frac{3}{8}l^2x\right)$ Max $y = -\frac{3}{320}\frac{Wl^3}{EI}$ at B $\theta = -\frac{1}{32}\frac{Wl^2}{EI}$ at A; $\theta = +\frac{1}{32}\frac{Wl^2}{EI}$ at B
19. End supports, end couple 	$R_1 = -\frac{M_0}{l}$ $R_2 = +\frac{M_0}{l}$ $V = R_1$	$M = M_0 + R_1x$ Max $M = M_0$ at A	$y = \frac{1}{6}\frac{M_0}{EI}\left(3x^2 - \frac{x^3}{l} - 2lx\right)$ Max $y = -0.0642\frac{M_0l^2}{EI}$ at $x = 0.422l$ $\theta = -\frac{1}{3}\frac{M_0l}{EI}$ at A; $\theta = +\frac{1}{6}\frac{M_0l}{EI}$ at B
20. End supports, intermediate couple 	$R_1 = -\frac{M_0}{l}$ $R_2 = +\frac{M_0}{l}$ (A to C) $V = R_1$	(A to B) $M = R_1x$ (B to C) $M = R_1x + M_0$ Max $-M = R_1a$ just left of B Max $+M = R_1a + M_0$ just right of B	(A to B) $y = \frac{1}{6}\frac{M_0}{EI}\left[\left(6a - 3\frac{a^2}{l} - 2l\right)x - \frac{x^3}{l}\right]$ (B to C) $y = \frac{1}{6}\frac{M_0}{EI}\left[3a^2 + 3x^2 - \frac{x^3}{l} - \left(2l + 3\frac{a^2}{l}\right)x\right]$ $\theta = -\frac{1}{6}\frac{M_0}{EI}\left(2l - 6a + 3\frac{a^2}{l}\right)$ at A; $\theta = +\frac{1}{6}\frac{M_0l}{EI}\left(1 - 3\frac{a^2}{l^2}\right)$ at C $\theta = \frac{M_0}{EI}\left(a - \frac{a^2}{l} - \frac{l}{3}\right)$ at B

21. Same spring in roll with stab. bar fixing one end.

TABLE III.—SHEAR, MOMENT, AND DEFLECTION FORMULAS FOR BEAMS.—(*Continued*)
Statically Indeterminate Cases

Loading, support, and reference number	Reactions R_1 and R_2, constraining moments M_1 and M_2, and vertical shear V	Bending moment M and maximum positive and negative bending moments	Deflection y, maximum deflection, and end slope θ
21. One end fixed, one end supported. Center load 	$R_1 = \frac{5}{16}W$ $R_2 = \frac{11}{16}W$ $M_2 = \frac{3}{16}Wl$ (A to B) $V = +\frac{5}{16}W$ (B to C) $V = -\frac{11}{16}W$	(A to B) $M = \frac{5}{16}Wx$ (B to C) $M = W(\frac{1}{2}l - \frac{11}{16}x)$ Max $+M = \frac{5}{32}Wl$ at B Max $-M = -\frac{3}{16}Wl$ at C	(A to B) $y = \frac{1}{96}\frac{W}{EI}(5x^3 - 3l^2x)$ (B to C) $y = \frac{1}{96}\frac{W}{EI}\left[5x^3 - 16\left(x - \frac{l}{2}\right)^3 - 3l^2x\right]$ Max $y = -0.00932\frac{Wl^3}{EI}$ at $x = 0.4472l$ $\theta = -\frac{1}{32}\frac{Wl^2}{EI}$ at A
22. One end fixed, one end supported. Intermediate load 	$R_1 = \frac{1}{2}W\left(\frac{3a^2l - a^3}{l^3}\right)$ $R_2 = W - R_1$ $M_2 = \frac{1}{2}W\left(\frac{a^3 + 2al^2 - 3a^2l}{l^2}\right)$ (A to B) $V = +R_1$ (B to C) $V = R_1 - W$	(A to B) $M = R_1x$ (B to C) $M = R_1x - W(x - l + a)$ Max $+M = R_1(l - a)$ at B; max possible value $= 0.174\,Wl$ when $a = 0.634l$ Max $-M = -M_2$ at C; max possible value $= -0.1927\,Wl$ when $a = 0.4227l$	(A to B) $y = \frac{1}{6EI}[R_1(x^3 - 3l^2x) + 3Wa^2x]$ (B to C) $y = \frac{1}{6EI}[R_1(x^3 - 3l^2x) + W(3a^2x - (x - l + a)^3)]$ If $a < 0.586l$, max y is between A and B at: $x = l\sqrt{1 - \frac{2l}{3l - a}}$ If $a > 0.586l$, max y is at $x = \frac{l(l^2 + a^2)}{3l^2 - a^2}$ If $a = 0.586l$, max y is at B and $= -0.0098\frac{Wl^3}{EI}$, max possible deflection $\theta = \frac{1}{4}\frac{W}{EI}\left(\frac{a^3}{l} - a^2\right)$ at A
23. One end fixed, one end supported. Uniform load 	$R_1 = \frac{3}{8}W$ $R_2 = \frac{5}{8}W$ $M_2 = \frac{1}{8}Wl$ $V = W\left(\frac{3}{8} - \frac{x}{l}\right)$	$M = W\left(\frac{3}{8}x - \frac{1}{2}\frac{x^2}{l}\right)$ Max $+M = \frac{9}{128}Wl$ at $x = \frac{3}{8}l$ Max $-M = -\frac{1}{8}Wl$ at B	$y = \frac{1}{48}\frac{W}{EIl}(3lx^3 - 2x^4 - l^3x)$ Max $y = -0.0054\frac{Wl^3}{EI}$ at $x = 0.4215l$ $\theta = -\frac{1}{48}\frac{Wl^2}{EI}$ at A

deflection

As can be seen defl^n. of 21 = 0·00932 × constant
against 11 = $\frac{1}{48}$ = ·0208 × "

∴ 11 deflects much more. ∴ 21 is stiffer.
by ·208 ...... = 2.235 ...